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BOUNDARY-VALUE PROBLEM IN THE MECHANICS OF THE DEFORMATION AND FAILURE OF DAMAGED BODIES WITH YIELDED ZONES*

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Traditionally, it is assumed that a deformable body yields under load at the moment when the stress-state or the stored energy parameters at the most critical points in the body reach the maximum allowable values and that the moment of failure is uniquely determined by the strength constants of the material. However, there is experimental evidence that the rigidity of the loading system also affects the resistance to failure. Here the loading system includes both the loading device (the testing machine, the structural elements which transmit the load, the working fluid or gas, etc.) and area of the deformable body around the failure zone [1, 2].

If the load is "soft," i.e., if the loading forces are independent of the resistance of a body in a homogeneous stress state, then failure really does correspond to the maximum stresses. In the other limiting case (a "rigid" load), the boundary points are displaced by a given amount, and damage can accumulate in an equilibrium process, which is reflected by a descending section in the stress–strain diagram [3-9]. If the loading system has finite rigidity, the moment that load-carrying capability is lost can correspond to one point or another on the descending part of the stress–strain diagram. The material state corresponding to the highest point on the stress–strain diagram is the critical point, but - to be more precise - failure is the final rapid non-equilibrium stage of this process, and can be viewed as the result of the loss of stability of accumulated damage at a supercritical strain stage. Also, the concept of a supercritical strain stage allows the reserves of load-carrying capacity to be used in optimizing structure design.

A more precise calculation that uses the total strain diagram requires the formulation and solution of boundary-value problems that consider material yield [10-13], and also possible stability losses in the weakened zones [4, 13, 14]. Here we present new boundary conditions that consider the rigidity of the loading system, formulate the defining equations, introduce supercritical strain conditions, obtain stability criteria for damage accumulation at the supercritical strain stage for an elementary material particle, and give a formulation of boundary-value problems that considers these effects within the framework of the mechanics of deformable solids.

1. Equation of State. For a material with microdamage, the stress tensor σ is related to the strain tensor ε in terms of a fourth-order damage-vulnerability tensor Ω by a defining equation in the form [15]

$$\sigma_{ij} = C_{ijmn}(I_{mnkl} - \Omega_{mnkl})\varepsilon_{kl}, \tag{1.1}$$

where C is the elastic modulus tensor; the $I_{klmn} = (1/2)(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm})$ are the components of the unit tensor, and δ_{kn} is the Kronecker delta.

In this model, all processes that change the material state are described by the damage-vulnerability tensor operator Ω, whose components are uniquely defined by the strain (loading) process. If the stresses can be defined by knowing the strains only at the current moment of time, then Ω is a function. When experimental data are reduced, this function can also describe the descending section of the stress–strain diagram.

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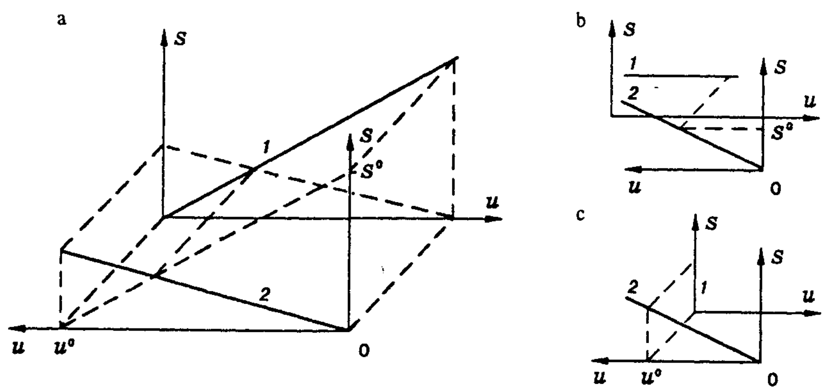


Fig. 1

Like the elastic modulus tensor, Ω has several independent components (their number and structure depend on the anisotropy of the damage accumulation processes), which define the other nonzero components. For example, for an initially isotropic material that remains isotropic as damage accumulates, the components of the Ω tensor are defined by two independent scalar quantities, and the equation of state has the form

$$\sigma_{ij} = [3K(1-æ)V_{ijmn} + 2G(1-g)D_{ijmn}]\varepsilon_{mn}.$$

The damage function æ and g depend on two invariants of the strain tensor:

$$j_\varepsilon^{(1)} = \varepsilon_{kk}, \quad j_\varepsilon^{(2)} = \sqrt{\bar{\varepsilon}_{ij}\bar{\varepsilon}_{ij}}, \quad \bar{\varepsilon}_{ij} = \varepsilon_{ij} - (1/3)\varepsilon_{kk}\delta_{ij}.$$

For an inelastic transverse-isotropic material, the defining equations that use the engineering elastic constants are written as

$$\begin{aligned}
&(1/2)(\sigma_{11}+\sigma_{22}) = k(1-æ)(\varepsilon_{11}+\varepsilon_{22}) + l(1-\varphi)\varepsilon_{33},\\
&(1/2)(\sigma_{11}-\sigma_{22}) = G_\perp(1-\rho_\perp)(\varepsilon_{11}-\varepsilon_{22}),\\
&\sigma_{33} = l(1-\varphi)(\varepsilon_{11}+\varepsilon_{22}) + n(1-\xi)\varepsilon_{33},\\
&\sigma_{12} = 2G_\perp(1-\rho_\perp)\varepsilon_{12}, \quad \sigma_{13} = 2G_{\parallel}(1-\rho_{\parallel})\varepsilon_{13}.
\end{aligned}$$

All the tensor components are defined by five independent functions: æ, φ, ξ, $\rho_\perp$, and $\rho_\parallel$. In the general case, the arguments of these functions are the four invariants of the strain tensor [16]:

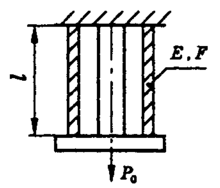
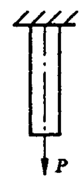
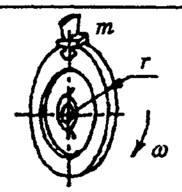
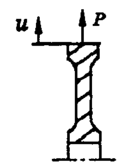
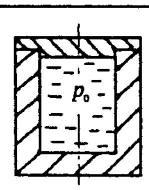
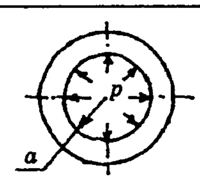
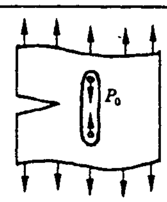
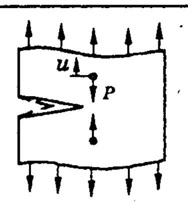
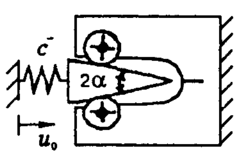
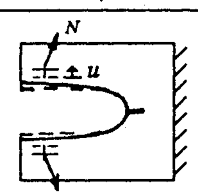
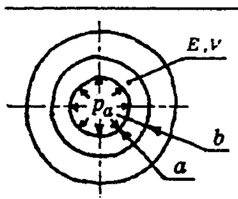
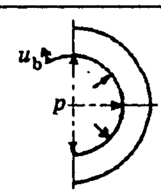
$$j_\varepsilon^{(1)} = \varepsilon_{11}+\varepsilon_{22}, \quad j_\varepsilon^{(2)} = \varepsilon_{33},$$
$$j_\varepsilon^{(3)} = \sqrt{(\varepsilon_{11}-\varepsilon_{22})^2 + 4\varepsilon_{12}^2}, \quad j_\varepsilon^{(4)} = \sqrt{\varepsilon_{13}^2+\varepsilon_{23}^2}.$$

For an orthotropic material, Eqs. (1.1) have the form

$$\begin{aligned}
&\sigma_{11} = C_{1111}(1-\lambda_1)\varepsilon_{11} + C_{1122}(1-\lambda_4)\varepsilon_{22} + C_{1133}(1-\lambda_6)\varepsilon_{33},\\
&\sigma_{22} = C_{1122}(1-\lambda_4)\varepsilon_{11} + C_{2222}(1-\lambda_2)\varepsilon_{22} + C_{2233}(1-\lambda_5)\varepsilon_{33},\\
&\sigma_{33} = C_{1133}(1-\lambda_6)\varepsilon_{11} + C_{2233}(1-\lambda_5)\varepsilon_{22} + C_{3333}(1-\lambda_3)\varepsilon_{33},\\
&\sigma_{12} = 2C_{1212}(1-\lambda_7)\varepsilon_{12}, \quad \sigma_{23} = 2C_{2323}(1-\lambda_8)\varepsilon_{23},\\
&\sigma_{13} = 2C_{1313}(1-\lambda_9)\varepsilon_{13}
\end{aligned}$$

where the arguments of the material functions λ_α are the six invariants of the strain tensor [16].

TABLE 1. Methods of Loading Various Bodies and the Corresponding Boundary Conditions

Body	Calculational diagram	Boundary conditions
		$P = P_0 - \frac{EF}{l} u$ F = cross-sectional area E = elastic modulus u = displacement of the end of the rod
		$P = m\omega^2 r - Ru$ Negative loading rigidity $R = -m\omega^2$
		$p = p_0 - \frac{2K}{a} u$ K = bulk compression modulus of the working fluid u = displacement of the inner wall of the cylinder
		$P = P_0 + 2Ru$ R = rigidity of the plate u = displacement of the rivet due to crack growth
		$N = \frac{c}{2\sin\alpha} u_0 - Ru,$ $R = \frac{c\cos\alpha}{2\sin^2\alpha}$ u = displacement of the bearing axis
		$p = \frac{2a^2}{a^2 + b^2 - \nu(b^2 - a^2)} p_a - Ru_b,$ $R = \frac{E(b^2 - a^2)}{b[a^2 + b^2 - \nu(b^2 - a^2)]}$

If failure zones can arise in a body during deformation, the boundary problem can be formulated more easily if the defining equations are written in the form [17]

$$\sigma_{ij} = C_{ijkl}(I_{klmn} - \Omega_{klmn})(I_{mnpq} - P_{mnpq})\varepsilon_{pq}$$

and the stepwise change in the deformation properties of the material are considered explicitly by using an indicator tensor **P**. The components of this tensor change stepwise from zero to unity when corresponding strength or stability conditions are no longer fulfilled for a supercritical strain and reflect that the material has lost the ability to resist a given type of load.

2. Boundary Conditions. We now formulate boundary conditions that consider the finite rigidity of the loading device. Let applied external forces on a section Γ_S of the boundary of the body be specified as follows:

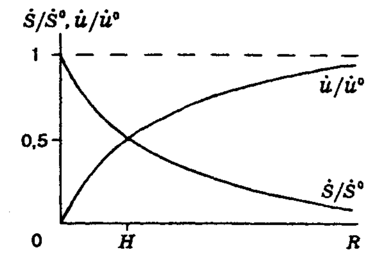


Fig. 2

$$[S_i(t) + R_{ij}u_j(t)]\Big|_{\Gamma_S} = S_i^0(t), \quad S_i = \sigma_{ij}n_j. \tag{2.1}$$

Here $\mathbf{S}^0(t)$ is the external force vector which is specified by the loading program; $\mathbf{u}$ is the displacement vector for points on the boundary with a normal $\mathbf{n}$; σ is the stress tensor; t is time; and $R_{ij}(\mathbf{u})$ is the rigidity of the loading device.

Here the boundary conditions explicitly include how the external loads change as the body deforms due to strain, a change usually neglected when the strains are small. However, just as small strains are studied in the mechanics of deformable solids because they can lead to large stresses, such small boundary displacements in a highly rigid loading machine deserve attention, because they can cause sharp changes in the external loads. This is especially true for boundary-value problems in which an energy balance is established during the formation of zones of damage accumulation, yielding, and failure. For example, engineering practice recognizes the substantial difference in the failure of hydraulic and pneumatic pressure vessels and piping. The way boundary-value problems are usually formulated, these cases are equivalent. The proposed boundary conditions augment the problem with information on the properties of the loading device and can describe the redistribution of energy between it and the deformable body.

We write the kinematic boundary conditions as in (2.1). Let the displacements on the boundary Γ_u be given as

$$[u_i(t) + Q_{ij}S_j(t)]\Big|_{\Gamma_u} = u_i^0(t), \tag{2.2}$$

where $Q_{ij}(\mathbf{S})$ is the compliance of the loading device and $\mathbf{u}^0$ is specified by the loading program. The specified forces and displacements are nominally related by the equations

$$S_i^0 = R_{ij}u_j^0, \quad u_i^0 = Q_{ij}S_j^0, \quad R_{ik}Q_{kj} = \delta_{ij}, \tag{2.3}$$

and Eqs. (2.1) and (2.2) are mutually invertible.

If $R_{ij} = 0$ or $Q_{ij} = 0$, the boundary conditions correspond to "soft" or "rigid" loading conditions, respectively, and formally coincide with the boundary conditions that are used in the mechanics of a deformable solid.

The interaction of the deformable body and the loading system is illustrated schematically in Fig. 1 (*a* is the general one-dimensional case, *b* is an absolutely "soft" loading, *c* is an absolutely "rigid" loading, *1* is the characteristic curve of the loading system, and *2* is the characteristic curve of the deformable body).

Several different boundary conditions that consider the rigidity of the loading device are shown for various technical systems in Table 1. In particular we note that in several cases the loading rigidity can be negative and that the rigidity of the loading device can be increased by special techniques, by using a wedge or plates.

The new formulation of the boundary conditions allow consideration of the difference between the real loading process or deformation and that nominally specified. An important characteristic of the loading method is its rate. We now show that the actual loading rate differs from the nominal, depending on the rigidity of the loading device.

Assume that the boundary-value problem is solved and that the relationship between forces and displacements is known at any point on the boundary of the body:

$$S_i = H_{ij}u_j, \quad u_i = P_{ij}S_j, \quad H_{ij}P_{jk} = \delta_{ik}.$$

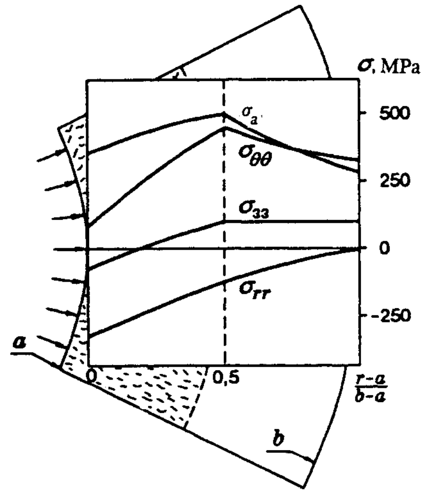


Fig. 3

Here the H_{ij} and P_{ij} are the rigidities and compliances of the deformable system. From these equations it follows from Eqs. (2.1)-(2.3) that

$$S_i = (\delta_{ik} + R_{ij}P_{jk})^{-1}S_k^0, \quad u_i = (\delta_{ik} + Q_{ij}H_{jk})^{-1}u_k^0.$$

If the properties of the loading and deformable systems do not depend on time, then the obvious equations relating the rates of change of the forces and the kinematic quantities remain valid ($\dot{S}_i^0 = dS_i^0/dt$, $\dot{S}_i = dS_i/dt$, $\dot{u}_i^0 = du_i^0/dt$, and $\dot{u}_i = du_i/dt$):

$$\dot{S}_i = (\delta_{ik} + R_{ij}P_{jk})^{-1}\dot{S}_k^0, \quad \dot{u}_i = (\delta_{ik} + Q_{ij}H_{jk})^{-1}\dot{u}_k^0.$$

The last equations are illustrated in Fig. 2 for the one-dimensional case. The higher the rigidity of the loading system, the closer the loading conditions are to $\mathbf{u}^0(t)$. The higher the compliance of the loading system, the closer the loading conditions are to $\mathbf{S}^0(t)$.

Thus, if the loading rigidity is nonzero, then the loading rate differs from the nominal, but if the loading compliance is nonzero, the displacement rate of the boundary points differs from the nominal.

3. Critical Strain Conditions. The traditional strength criteria, which are based on comparing the value of some function of stress or strain tensor components with its limiting value, usually do not include the rigidity of the loading system and correspond to zero rigidity. In this case, similar criteria can be used to estimate the critical stress state. We will characterize the limiting state of the material by combining two conditions: supercritical strain and the destabilization of the process. We now examine the first condition and consider the possibility of failure due to different mechanisms.

A phenomenological estimate of the failure of a solid body based on a strength criterion generally says nothing about the nature of the processes that led to the loss of load-bearing capability, although some criteria can have a physical interpretation. By using a combination of criteria, we can differentiate between failure mechanisms within the framework of a phenomenological approach.

Measures of the damage-vulnerability tensor $M_m(\mathbf{\Omega})$, which can be called measures of damage and which are functions of the components of $\mathbf{\Omega}$, can be used to construct criteria for the supercritical strain stage of isotropic and anisotropic materials. Let the corresponding constants of the critical damage Ω_m^* of a material be such that for any m a particle is intact if

$$M_m(\mathbf{\Omega}) < \Omega_m^*, \qquad m = 1, 2, \ldots \leqslant n$$

(n is the number of independent components of the tensor $\mathbf{\Omega}$), but can have a failure of type k if for some $m = k$

$$M_k(\mathbf{\Omega}) \geqslant \Omega_k^*. \tag{3.1}$$

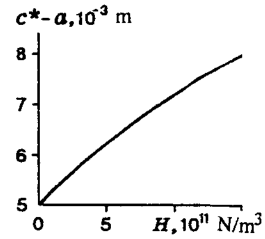


Fig. 4

If we want to consider even two different types of failure (from rupture and from shear, for example), we must examine at least two measures of damage.

For inelastic hard materials under a one-time loading, critical strain criterion (3.1) is equivalent to one of the inequalities

$$f_m(j_\varepsilon^{(1)},\ldots,j_\varepsilon^{(n)}) \geqslant C_m, \qquad m = 1, 2, \ldots \leqslant n, \tag{3.2}$$

where the $j_\varepsilon^{(n)}$ are independent invariants of the strain tensor; the C_m are material constants; and the f_m are universal functions.

Because the stress state, especially for composites, is nonhomogeneous, zones arise within the deformable body that to not satisfy the strength criteria. The question of whether or not failure of some microparticle leads to failure of the deformable body can be answered only after describing the stress redistribution process and possibly also the resultant damage of neighboring particles. Naturally this requires having data and making assumptions on the properties of a material particle which is damaged by some mechanism. Because of stress redistribution, it is possible that this particle can make a future contribution to resistance to an external load.

4. Stability Condition of the Supercritical Strain of a Material Point. The stability of the supercritical strain process, which is accompanied by an equilibrium growth of defects, can be analyzed by examining the relationship between energy outlays (the total growth of the elastic energy and the failure energy) and inputs (the work of external forces) for a hypothetically small growth in the supercritical strain. Within the framework of a phenomenological description, failure energy is taken as the energy dissipation related to the damage accumulation process. For an elementary material volume the failure energy and the increase in potential energy of elastic strain comprise the specific strain energy, which for any strain interval is the area under the equilibrium stress–strain curve. This curve is found experimentally by using a "rigid" test machine.

On the descending section of the stress–strain diagram, the breakup energy is greater than the strain energy. The larger this difference, the faster the curve drops to the final strain stage. Besides the external energy input, breakup process is also maintained by freeing the potential elastic strain energy.

Conceptually we remove from the body an elementary parallelepiped of volume $d\Omega$ in the neighborhood of the point in question. If we apply stresses σ' on the boundaries of the resultant void (the prime denotes the difference from stresses in the usual sense), they cause a strain ε.

We now establish a relationship between these stresses and strains:

$$\sigma'_{ij} = -V_{ijmn}\varepsilon_{mn}.$$

The tensor $\mathbf{V}$ can be called the effective rigidity tensor of the loading system and characterizes the strain properties of both the entire body and the of a loading device that provides specified displacements or forces on the boundary.

By using the effective rigidity tensor of the loading system, the work of external forces to make a virtual increase in the supercritical strain of a region Ω_0 with a boundary Γ can be represented by the equation

$$\delta A = \int\limits_{\Gamma} (\sigma_{ij}^0 - \frac{1}{2} V_{ijmn}\delta\varepsilon_{mn})\delta u_i n_j \, d\Gamma, \tag{4.1}$$

where the σ_{ij}^0 are stresses up to the variation in the strains. After we take the Gauss–Ostrogradskiĭ transform of (4.1), we have

$$\delta A = \int_{\Omega_0} (\sigma_{ij}^0 - \frac{1}{2} V_{ijmn} \delta\varepsilon_{mn}) \delta\varepsilon_{ij} \, d\Omega.$$

By comparing this equation with the formula for the strain energy

$$\delta E = \int_{\Omega_0} (\sigma_{ij}^0 - \frac{1}{2} D_{ijmn} \delta\varepsilon_{mn}) \delta\varepsilon_{ij} \, d\Omega,$$

which includes the tangential modulus tensor **D** at the supercritical stage of material strain:

$$d\boldsymbol{\sigma} = -\mathbf{D} : d\boldsymbol{\varepsilon},$$

we obtain that the stable state corresponds to the condition

$$\delta E - \delta A = \frac{1}{2} \int_{\Omega_0} (V_{ijmn} - D_{ijmn}) \delta\varepsilon_{mn} \delta\varepsilon_{ij} \, d\Omega > 0.$$

If we go to the elementary volume $d\Omega$ and introduce a comparative loading rigidity tensor **S**:

$$S_{ijmn} = V_{ijmn} - D_{ijmn}, \tag{4.2}$$

then the stability criterion for the supercritical strain of an elementary material particle in a body of finite dimensions will be equivalent to the requirement that the square of the tensor

$$S_{ijmn} \delta\varepsilon_{mn} \delta\varepsilon_{ij} > 0 \tag{4.3}$$

be positive.

We now find the rigidity tensor of the loading system. We write the equation for the work of external forces in the form

$$\delta A = \int_{\Gamma} \left(\sigma_{ij}^0 - \frac{1}{2} \frac{\partial \sigma_{ij}}{\partial u_k} \delta u_k\right) \delta u_i n_j \, d\Gamma. \tag{4.4}$$

By comparing (4.4) and (4.1) it follows that the components of the desired tensor can be found from the formula

$$V_{ijmn} = \frac{1}{2} \frac{\partial \sigma_{ij}}{\partial u_k} (dx_m \delta_{kn} + dx_n \delta_{km}). \tag{4.5}$$

As we see, the rigidity of the loading system depends on the relationship between the internal forces and displacements. Naturally, this is because the displacement of any point of the deformable body is determined by the strain of all its material particles, as well as the displacement of the boundaries, and in this sense it is an integral quantity that characterizes the rigidity of the loading system.

The relationship between the internal forces and displacements reflects the rigidity characteristics of all the material particles and elements of the loading system as a whole. Accordingly, the concept of an equivalent rigidity of the loading system has been introduced [4], which relates how a particular point is displaced in the direction of a principal stress acting on an elementary area. The supercritical strain condition of a small but finite region follows from Eqs. (4.2), (4.3), and (4.5) [4]. However only if these equations are written with respect to the principal axes can the partial derivatives in (4.5) be replaced by a ratio of absolute quantities and can dx be replaced by Δx such that each component of the S tensor is positive [this condition is sufficient but not necessary to fulfill (4.3)].

5. Boundary-Value Problem. By using the above approach, a boundary-value problem can be used to described the whole process of a quasistatic load deforming a body and causing it to fail, including the accompanying appearance and development of 1) damage zones, 2) material failure zones, and 3) regions of supercritical strain whose behavior is reflected on as a descending section on the stress–strain diagram. The boundary-value problem consists of a closed system of equations

$$\sigma_{ij,j}(t) = 0, \quad \varepsilon_{ij}(t) = (1/2)[u_{i,j}(t) - u_{j,i}(t)],$$
$$\sigma_{ij}(t) = C_{ijkl}[I_{klmn} - \Omega_{klmn}][I_{mnpq} - P_{mnpq}]\varepsilon_{pq}(t),$$
$$\Omega_{klmn} = \Omega_{klmn}(j_e^{(n)}, \dot{\varepsilon}, t), \quad P_{klmn} = P_{klmn}(f_k, C_k, \mathrm{S}),$$

along with the critical strain conditions (3.2), the stability condition of the critical strain (4.3), and the boundary conditions (2.1) and (2.2).

This formulation of the boundary-value problem of the continuum mechanics of failure is based on formulations of the boundary-value problem in the theories of elasticity and plasticity, but its main feature is that the defining equations include 1) the possibility of a descending section on the stress–strain diagram and 2) the use of stability criteria for supercritical strain processes. Naturally, the actual form of the equations can differ from those shown here.

As an illustration, we now examine the solution of the boundary-value problem for the supercritical strain of a thick-walled cylinder, which is loaded by an internal pressure. It is assumed that the stress–strain diagram of the material allows piecewise linear approximation with a shear modulus G in the elastic part and a softening modulus $-G_s$ in the section of supercritical strain. The critical stress state is reached when the absolute value of the stresses σ_a reaches a limiting value σ_{ab}. In order to include the rigidity $H = 2K/a$ of the loading system, which in this case includes the pressurizing device and the working liquid or gas, we use the boundary conditions shown in Table 1. An analytical solution of this problem is shown in Fig. 3 for the following data: $p = 326$ MPa, $a = 10$ mm, $b = 20$ mm, $\sigma_{ab} = 500$ MPa, $G = 2 \cdot 10^4$ MPa, $\nu = 0.3$, $G_s = 10^4$ MPa, and $H = 0$.

The calculations show tht the stable failure stage, when the absolute stress on the inner wall exceeds the yield stress, starts at a pressure of 216 MPa. The stable supercritical strain, which corresponds to "motion" along the descending part of the stress–strain diagram of points that reflect the stress–strain state of material particles on the edges of a zone with weakened bonds, continues as long as the outer radius of this zone does not exceed $c^* = 15$ mm and the pressure does not exceed 326 MPa. As we see, the reserve load capacity, observed with a refined calculation that includes the whole stress–strain diagram, is 51% in this case.

The nonzero rigidity of the loading system allows the failure process to be stabilized and increases the limiting dimension of the weakened zone ($c^* - a$). A graph of this function is shown in Fig. 4. The value of the parameter H calculated on the basis of [18] is $2 \cdot 10^7$ N/m^3 for air and $4.5 \cdot 10^{11}$ N/m^3 for water.

Just as in the analysis of elastic-plastic problems [19], the existence and uniqueness of the solution of the boundary problem in this case requires proof. The absence of a solution of the problem in the mathematical sense shows the impossibility of equilibrium resistance of a body with externally applied loads; i.e., of macrofailure.

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